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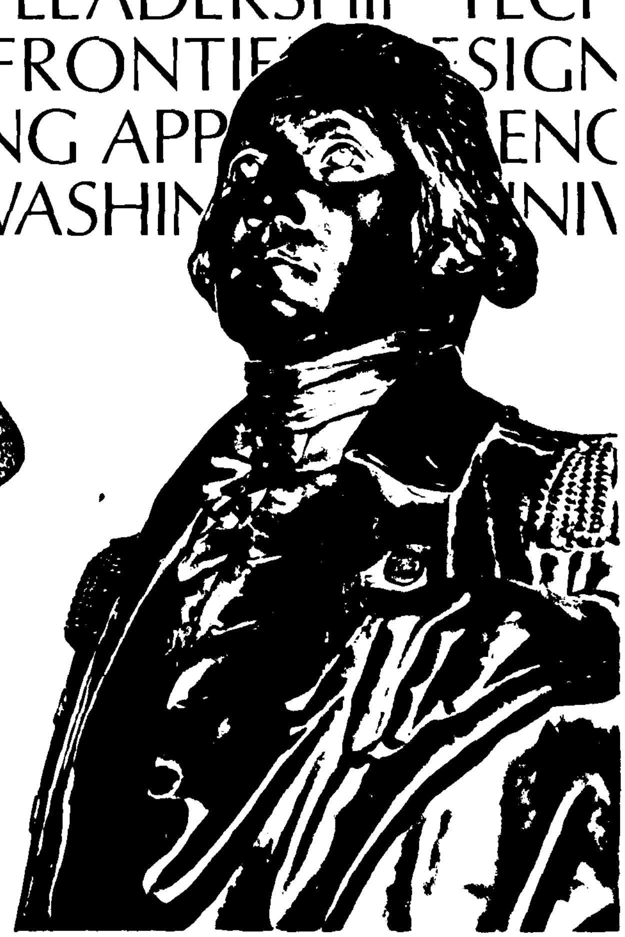

SCHOOL OF ENGINEERING
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CONTINUITY OF THE OPTIMAL VALUE FUNCTION UNDER THE MANGASARIAN-FROMOVITZ CONSTRAINT QUALIFICATION

by

Anthony V. Fiacco

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CONTINUITY OF THE OPTIMAL VALUE FUNCTION UNDER THE MANGASARIAN-FROMOVITZ CONSTRAINT QUALIFICATION

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Anthony V. Fiacco

1. Introduction

The sensitivity of the optimal value function of a mathematical program to perturbations of the problem parameters has been addressed by a number of authors. Using point-to-set maps, Berge [1] derived conditions sufficient for the semicontinuity of the optimal value function for programs with constraint set perturbations, and provided a general framework for some of the earliest work on the variation of the "perturbation function," i.e., the optimal objective function value, with changes in a parameter appearing in the right-hand side of the constraints. Evans and Gould [2] gave conditions guaranteeing the continuity of the perturbation function when the constraints are functional inequalities. Greenberg and Pierskalla [6] extended the work of Evans and Gould to obtain results for general constraint perturbations and obtained some initial results for programs with equality constraints. In [8], Hogan established conditions sufficient for the continuity of the perturbation function of a convex program, and in [7] gave conditions implying the continuity of the optimal value function of a nonconvex program when a parameter appears in the objective function. Gauvin and Tolle [5] showed that the perturbation function is continuous when the

problem functions are differentiable and the Mangasarian-Fromovitz Constraint Qualification is satisfied at a solution point.

We show that the optimal value function is continuous under weaker conditions than those previously invoked, for large classes of problems. The main purpose of this paper is to prove the continuity of the optimal value function of a general parametric nonlinear programming problem of the form

$$\min_x \; f(x,\varepsilon) \quad \text{s.t.} \quad g(x,\varepsilon) \geqq 0 \;, \quad h(x,\varepsilon) = 0 \;, \quad x \in C(\varepsilon) \subsetneq E^n \;, \quad P(\varepsilon)$$

when the Mangasarian-Fromovitz Constraint Qualification (MFCQ) is satisfied at a solution point. Here, $x \in E^n$ is the vector of decision variables, ε a parameter vector in E^k, and f, g_i, and h_j are real valued functions on $E^n \times E^k$, where $g(x,\varepsilon) = (g_1(x,\varepsilon),\ldots,g_m(x,\varepsilon))$ and $h(x,\varepsilon) = (h_1(x,\varepsilon),\ldots,h_p(x,\varepsilon))$. The set C, viewed as a function of ε, is a point-to-set map from E^k to subsets of E^n. Various continuity and differentiability assumptions will be invoked as needed.

The continuity of the optimal value function under the conditions that are assumed here was obtained by Fiacco for the inequality constrained problem [without the presence of the additional condition that $x \in C(\varepsilon)$] in [3]. Essentially simultaneously, Gauvin and Dubeau [4] independently obtained the continuity of the optimal value function under MFCQ for the inequality-equality constrained problem [without the presence of $C(\varepsilon)$]. Our results are slightly more general than those obtained in [4] and are obtained using a completely different approach.

2. Motivation and Preliminaries

The feasible region of problem $P(\varepsilon)$ will be denoted by $R(\varepsilon)$ and the set of solutions by $S(\varepsilon)$. The optimal value function is defined as $f^*(\varepsilon) \equiv \inf\{f(x,\varepsilon) \mid x \in R(\varepsilon)\}$. Thus, $S(\varepsilon) \equiv \{x \in R(\varepsilon) \mid f(x,\varepsilon) = f^*(\varepsilon)\}$. Again, when viewed as functions of ε, $R(\varepsilon)$ and $S(\varepsilon)$

define point-to-set maps from E^k to subsets of E^n . Continuity results for these maps will be obtained in the process of determining continuity properties of $f^*(\varepsilon)$.

The topological interior of a given set T will be denoted by T^0 .

If g_i $(i=1,\ldots,m)$ and h_j $(j=1,\ldots,p)$ are once continuously differentiable functions of x , then the Mangasarian-Fromovitz Constraint Qualification is said to hold at a point $x \in R(\varepsilon) \cap C^0(\varepsilon)$ if

(i) there exists a vector $\tilde{y} \in E^n$ such that

$$\begin{aligned} &\nabla_x g_i(x,\varepsilon)\tilde{y} > 0 \quad \text{for } i \text{ such that } g_i(x,\varepsilon) = 0 \text{ , and} \\ &\nabla_x h_j(x,\varepsilon)\tilde{y} = 0 \quad \text{for } j=1,\ldots,p \text{ ; and} \end{aligned} \tag{MFCQ}$$

(ii) the vectors $\nabla_x h_j(x,\varepsilon)$, $j=1,\ldots,p$, are linearly independent.

Here, the gradient with respect to x of a once differentiable function $F : E^n \times E^k \to E^1$ evaluated at (x,ε) is defined as the row vector $\nabla_x F(x,\varepsilon) \equiv (\partial F(x,\varepsilon)/\partial x_1,\ldots,\partial F(x,\varepsilon)/\partial x_n)$.

We also make use of the following continuity properties of real-valued functions and point-to-set maps. Related definitions and more detailed properties are developed in Berge [1] and Hogan [9].

Definition 2.1: Let ϕ be a real-valued function defined on the topological vector space X .

(i) ϕ is said to be lower semicontinuous (lsc) at a point $x_0 \in X$ if $\underline{\lim}_{x \to x_0} \phi(x) \geqq \phi(x_0)$.

(ii) ϕ is said to be upper semicontinuous (usc) at a point $x_0 \in X$ if $\overline{\lim}_{x \to x_0} \phi(x) \leqq \phi(x_0)$.

The notations $\underline{\lim}$ and $\overline{\lim}$ mean liminf and limsup, respectively. It is apparent that $\phi(x)$ is continuous at x_0 if and $\phi(x)$ is both usc and lsc at x_0.

Definition 2.2: Let $\phi : T \to Y$ be a point-to-set mapping from the topological vector space T to the topological vector space Y. Let $\{t_n\} \subset T$ be such that $t_n \to t_0$ in T as $n \to \infty$.

(i) ϕ is said to be open at a point t_0 of T if, for each $y_0 \in \phi(t_0)$, there exists a value m and a sequence $\{y_n\} \subset Y$ with $y_n \in \phi(t_n)$ for $n > m$ and $y_n \to y_0$.

(ii) ϕ is said to be closed at a point t_0 of T if $y_n \in \phi(t_n)$ and $y_n \to y_0$ together imply that $y_0 \in \phi(t_0)$.

If the mapping $\phi(t)$ is both open and closed at t_0, then $\phi(t)$ is said to be continuous at t_0.

The following notion will also be used in the subsequent results.

Definition 2.3: A point-to-set mapping $\phi : T \to Y$ is said to be uniformly compact near a point t_0 of T if the closure of the set $M = \bigcup_t \phi(t)$ is compact for t in some neighborhood $N(t_0)$ of t_0.

In obtaining continuity results for a solution of $P(\varepsilon)$, we shall first obtain corresponding continuity results for a problem of the same from as $P(\varepsilon)$ but without equality constraints,

$$\min_x \hat{f}(x,\varepsilon) \quad \text{s.t.} \quad \hat{g}(x,\varepsilon) \geqq 0 , \quad x \in \hat{C}(\varepsilon) \subseteq E^n . \qquad \hat{P}(\varepsilon)$$

There are several ways of reformulating problem $P(\varepsilon)$ into a locally equivalent problem that has the form $\hat{P}(\varepsilon)$. Two ways are by: (i) replacing $h(x,\varepsilon) = 0$ by $h(x,\varepsilon) \geqq 0$ and $-h(x,\varepsilon) \geqq 0$, mathematically

globally equivalent, in general, and (ii) eliminating the equalities by an appropriate elimination of variables, possible locally under appropriate assumptions, e.g., MFCQ. We make use of both devices in the sequel and obtain all the desired results for $P(\varepsilon)$ from the corresponding results for an equivalent problem of the form $\hat{P}(\varepsilon)$.

Without loss of generality, assume for simplicity that the particular value of ε of interest is $\varepsilon=0$. Suppose g and h are once continuously differentiable, $x^* \varepsilon R(0) \cap C^0(\varepsilon)$ for ε near 0, and MFCQ holds at $x^* \varepsilon R(0) \cap C^0(0)$. Then $h(x^*,0) = 0$ and, reordering variables if necessary, we may assume that $\nabla_{x_D} h(x^*,0)$ is nonsingular, where $x = (x_D, x_I)$, $x_D \varepsilon E^p$, $x_I \varepsilon E^{n-p}$, and $\nabla_{x_D} h$ is the Jacobian of h with respect to x_D. From the implicit function theorem, it is well known that there exist open sets $N^* \subseteq E^{n-p} \times E^k$ containing $(x_I^*,0)$ and $T^* \subseteq E^n \times E^k$ containing $(x^*,0)$ such that $h(x_D,x_I,0) = 0$ can be solved uniquely for x_D in terms of (x_I,ε) in N^*, and the function $x_D(x_I,\varepsilon)$ so defined is once continuously differentiable, in (x_I,ε), $(x_D(x_I,\varepsilon), x_I,\varepsilon) \varepsilon T^*$ and $x_D^* = x_D(x_I^*,0)$. Clearly, there exists an open set $C^* \subseteq N^*$ and containing $(x_I^*,0)$ such that $(x_D(x_I,\varepsilon), x_I) \varepsilon C(\varepsilon)$ if $(x_I,\varepsilon) \varepsilon C^*$. Since $\tilde{h}(x_I,\varepsilon) \equiv h[x_D(x_I,\varepsilon), x_I, \varepsilon] \equiv 0$ in N^*, problem $P(\varepsilon)$ can be reduced to the *locally* equivalent problem

$$\min_{x_I} \tilde{f}(x_I,\varepsilon) \quad \text{s.t.} \quad \tilde{g}(x_I,\varepsilon) \geqq 0 , \quad (x_I,\varepsilon) \varepsilon N^* \cap C^* , \qquad \tilde{P}(\varepsilon)$$

where

$$\tilde{f}(x_I,\varepsilon) \equiv f[x_D(x_I,\varepsilon), x_I, \varepsilon] \quad \text{and} \quad \tilde{g}(x_I,\varepsilon) \equiv g[x_D(x_I,\varepsilon), x_I, \varepsilon] .$$

Corresponding to the notation for $P(\varepsilon)$, we denote the feasible region of $\tilde{P}(\varepsilon)$ by $\tilde{R}(\varepsilon)$, the solution set by $\tilde{S}(\varepsilon)$, and the optimal value function by $\tilde{f}^*(\varepsilon)$. Other corresponding problem constituents will be similarly denoted. The analogous notation will be used for $\hat{P}(\varepsilon)$.

Defining $\tilde{C}(\varepsilon) \equiv \{x_I \mid (x,\varepsilon) \in N^* \cap C^*\}$, it is readily apparent that this defines a point-to-set mapping from decision variable space to parameter space, and that problem $\tilde{P}(\varepsilon)$ has the same general structure as problem $\hat{P}(\varepsilon)$. Thus, a precise connection between $P(\varepsilon)$ and $\hat{P}(\varepsilon)$ can again be established and exploited to obtain results for $P(\varepsilon)$, once results for $\hat{P}(\varepsilon)$ [or $\tilde{P}(\varepsilon)$] are known. The following result for $\tilde{P}(\varepsilon)$ was obtained by Fiacco and Hutzler [3] and will be needed. The labeling $x = (x_D, x_I)$ such that $\nabla_{x_D} h$ is nonsingular when MFCQ holds at $(x^*,0)$ is assumed.

Lemma 2.4 [3, Lemma 3.2]*:* If g and h are once continuously differentiable in x at $(x^*,0)$, then MFCQ holds at $x^* \in R(0) \cap C^0(0)$ with $\tilde{y} = (\tilde{y}_D, \tilde{y}_I) \in E^n$ the associated vector, where $\tilde{y}_D \in E^p$ and $\tilde{y}_I \in E^{n-p}$, if and only if MFCQ holds at $x_I^* \in \tilde{R}(0)$ with vector $\tilde{y}_I$.

Thus, the MFCQ property at a feasible point of $P(\varepsilon)$ is inherited by a feasible point of $\tilde{P}(\varepsilon)$.

The next result is independent of the variable-reduction derivation of $\hat{P}(\varepsilon)$. It follows from [3, Theorem 3.3] and [3, Theorem 3.6].

Lemma 2.5: If MFCQ is satisfied at some $\bar{x} \in \hat{S}(0) \cap \hat{C}^0(0)$, then for any fixed $\delta > 0$ and every unit vector $z \in E^k$ there exists $\bar{\beta}(\delta) > 0$ and a vector $\bar{y}(z)$ such that $\hat{g}(\bar{x} + \beta(\bar{y} + \delta\hat{y}) , \beta z) > 0$ for all $\beta \in (0,\bar{\beta}]$, where $\hat{y}$ is the vector associated with MFCQ at $\bar{x}_I$.

Finally, we shall also make use of a problem derived from $\tilde{P}(\varepsilon)$ as defined above. Let M be a closed subset of $N^* \cap C^*$ such that the interior M^0 of M contains $(x_I^*,0)$. Denote by $\bar{P}(\varepsilon)$ the problem $\tilde{P}(\varepsilon)$ with M replacing $N^* \cap C^*$. We continue to use the convention

adopted earlier, i.e., constituents associated with $\bar{P}(\varepsilon)$ will be denoted as they were for $P(\varepsilon)$, adding a superbar. Thus, for our problem

$$\min_{x_I} \; \tilde{f}(x_I,\varepsilon) \quad \text{s.t.} \quad \tilde{g}(x_I,\varepsilon) \geqq 0 \;, \quad (x_I,\varepsilon) \in M \subset N^* \cap C^* \;, \qquad \bar{P}(\varepsilon)$$

the feasible region is denoted by $\bar{R}(\varepsilon)$, the solution set by $\bar{S}(\varepsilon)$, the optimal value by $\bar{f}^*(\varepsilon)$, etc.

3. Continuity Properties of $\hat{f}^*(\varepsilon)$, $\hat{R}(\varepsilon)$ and $\hat{S}(\varepsilon)$

We first give an obvious generalization of rather well known results to obtain continuity results for a problem of the form

$$\min_x \; \hat{f}(x,\varepsilon) \quad \text{s.t.} \quad \hat{g}(x,\varepsilon) \geqq 0 \;, \quad x \in \hat{C}(\varepsilon) \subseteq E^n \;, \qquad \hat{P}(\varepsilon)$$

where $\hat{C}$ is a point-to-set map from E^k to subsets of E^n . Results implying these may be found in Berge [1] and Hogan [7], [8]. For completeness and for the uninitiated, we provide a proof.

Lemma 3.1:

(i) If g_i $(i=1,\ldots,m)$ is usc on $E^n \times E^k$, $\hat{C}(\varepsilon)$ is a closed set for each ε near 0 and a closed point-to-set mapping at $\varepsilon = 0$, $\hat{R}(0) \neq \phi$ and $\hat{R}(\varepsilon) \neq \phi$ and uniformly compact for ε near 0, then $\hat{R}(\varepsilon)$ is a closed map at $\varepsilon = 0$.

(ii) If, additionally, $\hat{f}$ is lsc on $E^n \times E^k$, then $\hat{f}^*(\varepsilon)$ is lsc at $\varepsilon = 0$.

Proof: Let $\varepsilon_n \to 0$ and assume $x_n \in \hat{R}(\varepsilon_n)$. Then $\hat{g}(x_n,\varepsilon_n) \geqq 0$ and $x_n \in C(\varepsilon_n)$. By the uniform compactness of $\hat{R}(\varepsilon)$, there exists a convergent subsequence $\{x_{n_j}\}$ and a point $\bar{x} \in E^n$ such that $x_{n_j} \to \bar{x}$.

Since $\hat{g}$ is usc on $E^n \times E^k$, $0 \leq \hat{g}(x_{n_j}, \varepsilon_{n_j}) \leq \hat{g}(\bar{x}, 0)$, and since $C(\varepsilon)$ is a closed mapping at 0, $\bar{x} \in C(0)$. Hence, $\bar{x} \in \hat{R}(0)$ and thus $\hat{R}(\varepsilon)$ is closed at 0.

For large n, $\hat{R}(\varepsilon_n) \neq \phi$ and compact and, since $\hat{f}$ is lsc, $\hat{S}(\varepsilon_n) \neq \phi$ for large n. Hence, there exists $x_n \in \hat{S}(\varepsilon_n)$ for large n. Uniform compactness of $\hat{R}(\varepsilon)$ near 0 implies the existence of a subsequence $\{x_{n_j}\}$ and a point $\bar{x}$ such that $x_{n_j} \to \bar{x}$, and the fact that $\hat{R}(\varepsilon)$ is a closed map at 0 implies that $\bar{x} \in \hat{R}(0)$. Thus, since f is lsc on $E^n \times E^k$,

$$\underline{\lim}_{\varepsilon \to 0} \hat{f}^*(\varepsilon) = \underline{\lim}_{j \to \infty} \hat{f}^*(\varepsilon_{n_j}) = \underline{\lim}_{j \to \infty} \hat{f}(x_{n_j}, \varepsilon_{n_j}) \geq \hat{f}(\bar{x}, 0) \geq \hat{f}^*(0),$$

and hence $\hat{f}^*(\varepsilon)$ is lsc at $\varepsilon = 0$.

Before proceeding to give sufficient conditions for the continuity of $\hat{f}^*(\varepsilon)$ in terms of the constituents of the given problem functions, we give a necessary and sufficient characterization of the continuity of $\hat{f}^*(\varepsilon)$ in terms of the relevant point-to-set maps. The necessity part, assuming $\hat{f}(x,\varepsilon)$ does not depend on ε and is continuous, and without the presence of $C(\varepsilon)$, was previously noted by Greenberg and Pierskalla [6].

Lemma 3.2: Suppose that all the assumptions of Lemma 3.1 are satisfied.

(i) If $\hat{f}^*(\varepsilon)$ is continuous at 0, then $\hat{S}(\varepsilon)$ is a closed point-to-set map at $\varepsilon = 0$.

(ii) If $\hat{S}(\varepsilon)$ is closed at 0 and $\hat{f}(x,\varepsilon)$ is continuous, then $\hat{f}^*(\varepsilon)$ is continuous at 0.

Proof: We first prove necessity of the consequences when $\hat{f}^*(\varepsilon)$ is assumed continuous. The fact that $\hat{R}(\varepsilon)$ is a closed map at 0 was

proved in Lemma 3.1. Consider $\varepsilon_n \to 0$ $(n \to \infty)$ and $x_n \in \hat{S}(\varepsilon_n)$ such that $x_n \to x^*$. Since $\hat{R}(\varepsilon)$ is closed at 0, $x^* \in \hat{R}(0)$. Since $\hat{f}^*(\varepsilon)$ is continuous at 0, we have that

$$\lim_{n\to\infty} \hat{f}(x^*,0) = \hat{f}^*(0) .$$

On the other hand,

$$\underline{\lim}_{n\to\infty} \hat{f}^*(\varepsilon_n) = \underline{\lim}\ \hat{f}(x_n,\varepsilon_n) \geq \hat{f}(x^*,0) \geq \hat{f}^*(0) .$$

Hence, it follows that $\hat{f}(x^*,0) = \hat{f}^*(0)$ and $x^* \in \hat{S}(0)$, so $\hat{S}(\varepsilon)$ is also a closed map at 0 and Part (i) is proved.

The fact that $\hat{S}(\varepsilon)$ closed at 0 and $\hat{f}(x,\varepsilon)$ continuous implies continuity of $\hat{f}^*(\varepsilon)$ at 0 can be proved as follows. We must show that $\lim_{n\to\infty} \hat{f}^*(\varepsilon_n) = \hat{f}^*(0)$ whenever $\varepsilon_n \to 0$. The existence of $x_n \in \hat{S}(\varepsilon_n)$ for n large was indicated in proving Lemma 3.1. There must exist a convergent subsequence $\{x_{n_j}\}$ and a point x^* such that $x_{n_j} \to x^*$, since $\hat{R}(\varepsilon)$ is uniformly compact near 0. Since $\hat{S}(\varepsilon)$ is a closed map at 0, it follows that $x^* \in \hat{S}(0)$. Hence,

$$\lim_{j\to\infty} \hat{f}^*(\varepsilon_{n_j}) = \lim_{j\to\infty} \hat{f}(x_{n_j},\varepsilon_{n_j}) = \hat{f}(x^*,0) = \hat{f}^*(0) .$$

Since the same argument can be applied to any convergent subsequence of $\{x_n\}$, it follows that $\lim_{n\to\infty} \hat{f}^*(\varepsilon_n) = \hat{f}^*(0)$.

Recall that $\hat{C}^0$ denotes the topological interior of $\hat{C}$.

Theorem 3.3: If $\hat{f}(x,\varepsilon)$ is continuous, $\hat{g}(x,\varepsilon)$ is once continuously differentiable in x and $\hat{R}(\varepsilon)$ is uniformly compact for ε near 0, $\hat{C}(\varepsilon)$ is a closed subset of E^n for each ε near 0 and a closed point-to-set mapping at $\varepsilon = 0$, $\exists x^* \in \hat{S}(0) \cap \hat{C}^0(\varepsilon)$ for ε near 0, and if MFCQ holds at x^*, then $\hat{S}(\varepsilon) \neq \phi$ near $\varepsilon = 0$, $\hat{R}(\varepsilon)$ and $\hat{S}(\varepsilon)$ are closed maps at 0 and $\hat{f}^*(\varepsilon)$ is continuous at 0.

Proof: From Lemma 2.5 it follows that $\hat{R}(\varepsilon_n) \neq \phi$ for n sufficiently large, given any sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ as $n \to \infty$. Consequently, it follows that the assumptions of Lemma 3.1 are satisfied, and hence $\hat{R}(\varepsilon)$ is a closed map at $\varepsilon = 0$ and $\hat{f}^*(\varepsilon)$ is lsc at $\varepsilon = 0$. The fact that $\hat{S}(\varepsilon) \neq 0$ for ε near 0 is immediate and was noted in proving Lemma 3.1.

Now, let $\delta > 0$, let $\hat{y}$ be given by MFCQ for x^*. From Lemma 2.5, it follows that there exists a vector $\bar{y}$ such that $x^* + \beta_k(\bar{y} + \delta\hat{y}) \in \hat{R}(\beta_k z)$ for all unit vectors $z \in E^k$ provided that $\beta_k > 0$ is sufficiently near 0. Letting $\varepsilon_k \to 0$, and without loss of generality, assuming $\varepsilon_n \neq 0$ for all n, setting $\beta_k = \|\varepsilon_k\|$ and $z_k = \varepsilon_k/\|\varepsilon_k\|$, we see that

$$\overline{\lim_{\varepsilon\to 0}}\, \hat{f}^*(\varepsilon) = \overline{\lim_{\varepsilon_k \to 0}}\, \hat{f}^*(\varepsilon_k) \leqq \overline{\lim_{k\to\infty}}\, \hat{f}(x^* + \beta_k(\bar{y} + \delta\hat{y}) , \beta_k z_k) = \hat{f}(x^*,0) = \hat{f}^*(0) .$$

Thus, $\hat{f}^*(\varepsilon)$ is also usc at $\varepsilon = 0$ and we may conclude that $\hat{f}^*(\varepsilon)$ is continuous at $\varepsilon = 0$.

The fact that $\hat{S}(\varepsilon)$ is a closed mapping at $\varepsilon = 0$ was shown in Lemma 3.2, Part (i), and the proof is complete.

4. Continuity Properties of $f^*(\varepsilon)$, $R(\varepsilon)$, and $S(\varepsilon)$

We now use the results obtained for the inequality constrained problem $\hat{P}(\varepsilon)$ to obtain the analogous results for the problem also containing equality constraints,

$$\min_x \ f(x,\varepsilon) \quad \text{s.t.} \quad g(x,\varepsilon) \geqq 0 , \quad h(x,\varepsilon) = 0 , \quad x \in C(\varepsilon) \subsetneq E^n , \quad P(\varepsilon)$$

when $h(x,\varepsilon)$ is continuous.

The results of Lemmas 3.1 and 3.2 are applicable immediately by replacing $h(x,\varepsilon) = 0$ and $h(x,\varepsilon) \geqq 0$ and $-h(x,\varepsilon) \geqq 0$ to obtain an equivalent problem of the form $\hat{P}(\varepsilon)$. If h is assumed continuous in

$E^n \times E^k$, then h and $-h$ are usc in (x,ε) and the lemmas can be applied to the equivalent reformulated problem, noting that $f^*(\varepsilon)$, $R(\varepsilon)$, and $S(\varepsilon)$ remain unaltered. This proves the following corollaries of Lemmas 3.1 and 3.2 which, for convenience and conciseness, we collect and restate as a theorem, in terms of the constituents of Problem $P(\varepsilon)$.

Theorem 4.1:

(i) If g_i $(i=1,\ldots,m)$ is usc and h_j $(j=1,\ldots,p)$ is continuous on $E^n \times E^k$, $C(\varepsilon)$ is a closed set for each ε near 0 and a closed point-to-set mapping at $\varepsilon=0$, $R(0) \neq \phi$, and $R(\varepsilon) \neq \phi$ and uniformly compact for ε near 0, then $R(\varepsilon)$ is a closed map at $\varepsilon=0$.

(ii) If, additionally, f is lsc on $E^n \times E^k$, then $f^*(\varepsilon)$ is lsc at $\varepsilon=0$.

(iii) If the assumptions of Part (i) hold and $f^*(\varepsilon)$ is continuous at 0, then $S(\varepsilon)$ is a closed map at $\varepsilon=0$.

(iv) If the assumptions of Part (i) hold and $S(\varepsilon)$ is a closed map at $\varepsilon=0$ and $f(x,\varepsilon)$ is continuous, then $f^*(\varepsilon)$ is continuous at 0.

We are now in a position to prove our principal result, the precise analogy of Theorem 3.3, relative to the respective constituents of Problem $P(\varepsilon)$. Although ostensibly an extension of Theorem 3.3, we shall see that the result follows as a corollary.

Theorem 4.2: If $f(x,\varepsilon)$ is continuous, $g(x,\varepsilon)$ and $h(x,\varepsilon)$ are once continuously differentiable in x , $R(\varepsilon)$ is uniformly compact for ε near 0, $C(\varepsilon)$ is a closed subset of E^n for ε near 0 and a closed point-to-set mapping at $\varepsilon=0$, $\exists x^* \in S(0) \cap C^C(\varepsilon)$ for each ε near 0, and if MFCQ holds at x^* , then $S(\varepsilon) \neq \phi$ near $\varepsilon=0$, $R(\varepsilon)$ and $S(\varepsilon)$ are closed maps at $\varepsilon=0$, and $f^*(\varepsilon)$ is continuous at 0.

Proof: We eliminate the equalities of $P(\epsilon)$ at x^* for (x_I,ϵ) in a neighborhood N^* of $(x_I^*,0)$, where $x^* = (x_D^*,x_I^*)$, using the previously defined (see Section 2) variable reduction transformation, constructing a problem $P'(\epsilon)$ of the form $\tilde{P}(\epsilon)$, with (x_I,ϵ) restricted to N^* and the additional restriction that $(x_D(x_I,\epsilon),x_I) \in C(\epsilon)$. We know that $x_I^* \in \tilde{S}(0)$ and that MFCQ holds at x_I^* (Lemma 2.4).

We now formulate a problem of the form $\bar{P}(\epsilon)$, with the additional proviso that the closed subset M of N^*, such that $(x_I^*,0) \in M^0$, be selected so that $(x_D(x_I,\epsilon),x_I) \in C^0(\epsilon)$ for ϵ near 0 and for all (x_I,ϵ) in M. This can be done, since $x^* = (x_D(x_I^*,0),x_I^*) \in C^0(\epsilon)$ for ϵ near 0 (by assumption), $(x_I^*,0) \in M^0$, and $x_D(x_I,\epsilon)$ is continuous (by construction). Define $\hat{C}(\epsilon) \equiv \{x_I \mid (x_I,\epsilon) \text{ in } M\}$. Clearly, $\hat{C}(\epsilon)$ is a point-to-set map from E^k to subsets of E^{n-p}. Since M is a closed subset of $E^{n-p} \times E^k$, $\hat{C}(\epsilon)$ is a closed set and a closed point-to-set mapping for each ϵ in the domain of the mapping, including of course $\epsilon=0$. By construction, $x_I^* \in \hat{C}^0(0)$. Also, $x_I^* \in \hat{C}^0(\epsilon)$ for each ϵ near 0, since $(x_I^*,0) \in M^0$.

It follows that $\bar{P}(\epsilon)$ has the same structure as $\hat{P}(\epsilon)$ and we now show that the previous results are applicable. Clearly, the uniform compactness of $R(\epsilon)$ near $\epsilon=0$ implies the uniform compactness of $\bar{R}(\epsilon)$ near $\epsilon=0$. Also, $\tilde{f}(x_I,\epsilon) \equiv f(x_D(x_I,\epsilon),\epsilon)$ is continuous in M, since $x_D(x_I,\epsilon)$ is continuous in M. It remains only to show that $x_I^* \in \bar{S}(0)$, and the conditions of Theorem 3.3 will be satisfied by the respective constituents of $\bar{P}(\epsilon)$. Clearly, $f^*(\epsilon) \leq \tilde{f}^*(\epsilon) \leq \bar{f}^*(\epsilon)$ and since $f^*(0) = f(x^*,0) = f(x_D(x_I^*,0), x_I^*, 0) = \tilde{f}(x_I^*,0) = \bar{f}^*(0)$, we conclude that $\bar{f}^*(0) = \tilde{f}^*(0) = f^*(0)$, which also implies that $x_I^* \in \bar{S}(0)$.

The assumptions of Theorem 3.3 are satisfied relative to the respective constituents of $\bar{P}(\epsilon)$ and relative to the space E^{n-p}. We

conclude that $\bar{f}^*(\ )$ is continuous at $\epsilon=0$. These relationships imply that $\overline{\lim}_{\epsilon\to 0} f^*(\epsilon) \le \overline{\lim}_{\epsilon\to 0} \bar{f}^*(\epsilon) = \bar{f}^*(0) = f^*(0)$; i.e., $f^*(\epsilon)$ is usc at 0. Since $\bar{R}(\epsilon) \neq 0$ for ϵ near 0 (Lemma 2.5), it follows that $R(\epsilon) \neq 0$ for ϵ near 0. We conclude from Theorem 4.1, Part (ii), that $f^*(\epsilon)$ is lsc at $\epsilon=0$. Therefore, $f^*(\epsilon)$ is continuous at $\epsilon=0$. The remaining conclusions follow from Parts (i) and (iii) of Theorem 4.1.

By taking $C = E^n$, we obtain a realization of the theorem relative to the usual inequality-equality constrained parametric problem,

$$\min_x \ f(x,\epsilon) \quad \text{s.t.} \quad g(x,\epsilon) \ge 0\ , \quad h(x,\epsilon) = 0\ , \quad x \in E^n\ . \qquad P_1(\epsilon)$$

Corollary 4.3: If $f(x,\epsilon)$ is continuous, $g(x,\epsilon)$ and $h(x,\epsilon)$ are once continuously differentiable in x and $R(\epsilon)$ is uniformly compact for ϵ near 0, and if MFCQ holds at some $x^* \in S(0)$, then $S(\epsilon) \neq \emptyset$ near $\epsilon=0$, $R(\epsilon)$ and $S(\epsilon)$ are closed maps at $\epsilon=0$, and $f^*(\epsilon)$ is continuous at $\epsilon=0$.

5. Concluding Remarks, an Example, and Extensions

It should also be noted that the assumptions of Theorem 4.2 do not preclude the possibility that the set C by "truly binding" in the sense that the solution set may change if C is perturbed. Also, the assumptions do not imply that there exists $x^n \in S(\epsilon_n)$ such that $x^n \to x^*$ as $\epsilon_n \to 0$, or for that matter, such that $x^n \to \bar{x} \in S(0) \cap C^0(0)$. That is, it is possible that some or all the limit points as $\epsilon_n \to 0$ of a sequence $\{x^n\}$ of solutions of $R(\epsilon_n)$ [one of which must exist and all of which must be in the solution set $S(0)$ of $P(0)$] are not MFCQ points or may lie on the boundary of C. Thus, although the assumptions qualifying the behavior of the solution set $S(0)$ are made relative to $C^0(0)$, this does not mean that $C(\epsilon)$ need play a passive role near or at $\epsilon=0$. The following example verifies these assertions.

Consider the problem

$$P(\varepsilon): \min \quad x_2$$

$$\text{s.t.} \quad x_2 \geq \varepsilon_1 x_1 \; , \quad (x_1+1)^2 + x_2^2 \geq 0 \; , \quad x_2 \leq 1$$

$$x \in C(\varepsilon) \equiv C = \{x \in E^2 \mid -1 \leq x_1 \leq 1\} \; ,$$

$$\text{with} \quad \varepsilon \equiv \varepsilon_1 > 0 \; .$$

It is easy to see that the solution set of $P(\varepsilon)$ is $S(\varepsilon) = \{x(\varepsilon)\} = (-1, -\varepsilon_1)$, and the solution set of $P(0)$ is given by $S(0) = \{x \in C \mid x_2 = 0\}$. Note that $x^* \in \{x \in E^2 \mid -1 < x_1 < 1, \; x_2 = 0\}$ implies $x^* \in S(0) \cap C^0(\varepsilon)$ for every ε and MFCQ holds at any such x^*, relative to $R(0)$. The remaining assumptions of Theorem 4.2 obviously hold, for this example. Clearly, $x(\varepsilon) \to x(0) = (-1,0)$ as $\varepsilon \to 0$. But $x(0)$ is on the boundary of C. Also, MFCQ cannot be satisfied at $x(0)$ because the gradient of the second inequality constraint is 0 at $x(0)$. Finally, it is apparent that $S(0)$ changes if C is perturbed, so the various possibilities indicated have been demonstrated.

It should also be observed that the same device used to obtain Theorem 4.1 [concerning Problem $P(\varepsilon)$] from Lemma 3.1 and Lemma 3.2 [concerning Problem $\hat{P}(\varepsilon)$], cannot be used to obtain Theorem 4.2 from Theorem 3.3. Recall that we simply replaced $h(x,\varepsilon) = 0$ by $h(x,\varepsilon) \geq 0$ and $-h(x,\varepsilon) \geq 0$ and the analogous result for $P(\varepsilon)$ was obtained immediately from that for $\hat{P}(\varepsilon)$. However, if we make this substitution in $P(\varepsilon)$ to obtain a problem of the form $\hat{P}(\varepsilon)$, we find that the resulting problem cannot satisfy MFCQ (since there can be no vector y simultaneously satisfying $\nabla_x h(x,\varepsilon)y > 0$ and $-\nabla_x h(x,\varepsilon)y > 0$, for any x or ε). Hence, the assumption regarding MFCQ in Theorem 3.3 cannot be satisfied and the theorem is inapplicable to proving Theorem 4.2 via this construct.

Finally, it is noted that the conditions regulating the behavior of the set $\hat{C}(\varepsilon)$ in Theorem 3.3 and $C(\varepsilon)$ in Theorem 4.2 can be

weakened. For example, let $\hat{R}_1(\varepsilon) \equiv \{x \varepsilon E^n \mid \hat{g}(x,\varepsilon) \geq 0\}$. Then, the condition that there exist $x^* \varepsilon \hat{S}(0) \cap \hat{C}^0(\varepsilon)$ for ε near 0, may be replaced by: there exists $x^* \varepsilon \hat{S}(0) \cap \hat{C}(0)$, x^* solves: min $\hat{f}(x,\varepsilon)$ s.t. $x \varepsilon \hat{R}_1(0)$, and for $\delta > 0$ and any unit vector z , there exists $\bar{y}(z)$ and $\bar{\beta}(\delta)$ such that $x^* + \beta(\bar{y} + \delta\hat{y})$ (as in Lemma 2.5) is in $C(\beta z)$ for $\beta \varepsilon (0,\hat{\beta}]$. This would yield as a corollary that the optimal value of min $f(x,\varepsilon)$ s.t. $x \varepsilon C(\varepsilon)$ is continuous. We have held to the given assumptions, both for simplicity and to provide fairly general sufficient conditions on the constituents of the given problem. It is mathematically desirable to require "no more" of $C(\varepsilon)$ than is required of $\hat{R}_1(\varepsilon)$. This leads to the idea of defining a constraint qualification (CQ) for a general set that is analogous to and compatible with MFCQ defined relative to constraints of the form $g \geq 0$ and $h = 0$. It might be fruitful to explore this possibility, along with the determination of other CQ's relative to general sets.

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